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The Least Root of a Continuous Function

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Abstract—For each $\varepsilon > 0$ and each scalar real valued and continuous on a compact set $\Omega \subset R^n$, $\xi \in [a, b]$ function $g(\tau, \xi)$ such that $g(\tau, a) \cdot g(\tau, b) < 0$ we construct a function $g_\varepsilon(\tau, \xi)$, for which the least root $\xi(\tau)$ of the equation $g_\varepsilon(\tau, \xi) = 0$ continuously depends on τ, while $|g(\tau, \xi) - g_\varepsilon(\tau, \xi)| < \varepsilon$. We give examples illustrating the fact that in a general case assumptions are unimprovable.

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1. INTRODUCTION

Let a function $\xi(\tau)$ be defined implicitly, namely, let it be a solution to the equation $g(\tau, \xi) = 0$. In applications, one often needs either to find this function or at least to prove its existence. If the function $g(\tau, \xi)$ is scalar, then the existence of $\xi(\tau)$ follows from the Bolzano theorem [1, 2]: *if real valued $g(\tau, \xi)$ with each τ is continuous on $[a, b]$ and $g(\tau, a) \cdot g(\tau, b) < 0$ then there exists $\xi(\tau) \in (a, b)$ such that $g(\tau, \xi(\tau)) = 0$.* However the Bolzano theorem does not answer the following question (which is very important in applications): in which cases the function $\xi(\tau)$ is continuous? One can also prove the existence of the function $\xi(\tau)$ with the help of the implicit function theorem (see, for example, [3. pp. 568–586]).

Assume that a function $F(u, x, y)$ is differentiable in some neighborhood of a point $M_0(\acute{u}, \acute{x}, \acute{y})$ in the space R, while the partial derivative $\partial F/\partial u$ is continuous at the point M_0. Then, if the function F vanishes at the point M_0, but the partial derivative $\partial F/\partial u$ does not vanish, then for any sufficiently small positive value ε there exists a neighborhood of $M_0'(\acute{x}, \acute{y})$ in the space R' such that one can find a function $u = \varphi(x, y)$ defined in this neighborhood which satisfies the condition $|u - \acute{u}| < \varepsilon$ and represents a solution to the equation $F(u, x, y) = 0$. Moreover, the function $u = \varphi(x, y)$ is continuous and differentiable in the mentioned neighborhood of the point $M_0'(\acute{x}, \acute{y})$.

This property guarantees the differentiability (and, consequently, the continuity) of the root $\xi(\tau)$ to the equation $g(\tau, \xi) = 0$ only in a small neighborhood of the point (τ_0, ξ_0), but not with all τ_0. We study the continuity of the solution in the case when $\xi(\tau)$ is the least root of the equation $g(\tau, \xi) = 0$. Note that the continuity of $\xi(\tau)$ does not depend on the smoothness of $g(\tau, \xi)$. Below we consider the corresponding

Example 1. Assume that a function $x(\tau) \in [a_1, b_1] \subset [a, b]$ is infinitely smooth, $\xi \in [a, b]$ and $|a|, |b|$ are sufficiently large. Introduce the function $g(\tau, \xi) = (\xi - x(\tau))^2(x(\tau) + 1 - \xi) + c(\tau)$, where $c(\tau)$ is infinitely smooth, $c(\tau_0) = 0$, $c(\tau) > 0$ with $\tau \neq \tau_0$, while $g(\tau, a) > 0$, $g(\tau, b) < 0$. Evidently that $x(\tau_0)$ is the least root of the equation $g(\tau_0, \xi) = 0$. However with sufficiently small $|\tau - \tau_0|$, $\tau \neq \tau_0$, the least root equals $x(\tau_0) + 1 + \delta$, and $\delta > 0$. This, in turn, means that the least root of the equation $g(\tau, \xi) = 0$ is discontinuous at the point τ_0.

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